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Drawing the complex projective structures on once-punctured tori

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1 Introduction

This report is based on my talk at RIMS International Conference on "Geometry Related to Integrable Systems" organized by Reiko Miyaoka. In my talk I showed many interesting pictures of one-dimensional Teichmüller spaces and related spaces created by Yasushi Yamashita (Nara Women's Univ.) which were already appeared in [3]. In this report I would like to explain the background of these pictures, which are explained more extensively in [2]. I would like to thank Yasushi Yamashita for his kind assistance with computer graphics, and Yoshihiro Ohnita for his constant encouragement for me to write this report.

2 Definition of $T(X)$

Let X be a Riemann surface of genus g with n punctures. Here we assume that X is uniformized by the upper half plane $\mathbb{H}$ in $\mathbb{C}$, which implies the inequality $2g-2+n>0$. The *Teichmüller space* $T(X)$ of X is the set of equivalent classes of quasi-conformal homeomorphisms from X to other Riemann surface Y, $f: X \rightarrow Y$: two maps $f_1: X \rightarrow Y_1$ and $f_2: X \rightarrow Y_2$ are equivalent if $f_2 \circ f_1^{-1}: Y_1 \rightarrow Y_2$ is homotopic to a conformal map. If we assume $f: X \rightarrow Y$ as a quasi-conformal deformation of X, $T(X)$ can be considered as the space of quasi-conformal deformations of X.

We will consider a complex manifold structure on $T(X)$, embed it holomorphically into complex affine space and try to draw its figure. For this purpose, we give another characterization of $T(X)$ due to Ahlfors and Bers in the next section.

3 Complex structure on $T(X)$

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a Fuchsian group uniformizing $X = \mathbb{H}/\Gamma$. A measurable function $\nu(z)$ on the Riemann sphere $\mathbb{C}P^1$ whose essential sup norm is less than 1 is called a *Beltrami differential* for Γ if μ is equal to 0 on the lower half plane $\mathbb{L}$ in $\mathbb{C}$ and satisfies

$$\mu(\gamma(z)) \cdot \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$$

for all $z \in \mathbb{C}P^1$ and $\gamma \in \Gamma$. This functional equality implies that μ on $\mathbb{H}$ is a lift of $(-1, 1)$ form on X. We denote the set of Beltrami differentials by $B_1(\Gamma, \mathbb{H})$ which has a structure of a unit ball of complex Banach space. The measurable Riemann's mapping theorem due to Ahlfors and Bers guarantees that for any $\mu \in B_1(\Gamma, \mathbb{H})$ there exists a quasi-conformal map $f^\mu : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that f^μ satisfies the Beltrami equation

$$\frac{\partial f^\mu}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f^\mu}{\partial z}(z).$$

Also f^μ is unique up to post-composition by Möbius transformations.

Here we have two remarks: (i) f^μ is conformal on $\mathbb{L}$. (ii) The quasi-conformal conjugation of Γ by f^μ, $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$ is also a discrete subgroup of $PSL_2(\mathbb{C})$ acting conformally on $f^\mu(\mathbb{H})$.

Now we say $\mu_1 \sim \mu_2$ for $\mu_1, \mu_2 \in B_1(\Gamma, \mathbb{H})$ if $\Gamma^{\mu_1} = \Gamma^{\mu_2}$. Then $T(X)$ can be identified with the quotient space $B_1(\Gamma, \mathbb{H})/\sim$ as follows: For any $[\mu] \in B_1(\Gamma, \mathbb{H})/\sim$, we have a quasi-conformal deformation of X

$$f^\mu : X = \mathbb{H}/\Gamma \to f^\mu(\mathbb{H})/\Gamma^\mu$$

which defines a point of $T(X)$. $T(X)$ becomes a complex manifold of $dim_{\mathbb{C}} T(X) = 3g - 3 + n$ through the complex structure of $B_1(\Gamma, \mathbb{H})$. We will embed $T(X)$ holomorphically into the complex linear space by means of complex projective structures on $\bar{X}$, the mirror image of X which will be explained in the next section.

4 Complex projective structures on $\bar{X}$

Let S be a surface. A *complex projective structure*, so called $\mathbb{C}P^1$*-structure on* S is a maximal system of charts with transition maps in $PSL_2(\mathbb{C})$. Since elements of $PSL_2(\mathbb{C})$ are holomorphic, any $\mathbb{C}P^1$-structure on S determines its underlying complex structure. Suppose we consider a $\mathbb{C}P^1$-structure whose underlying complex structure is equal to $\bar{X} = \mathbb{L}/\Gamma$, the mirror image of X. For a local coordinate function of this $\mathbb{C}P^1$-structure, we can take its analytic continuation along any curve on $\bar{X}$ and have a multi-valued locally univalent holomorphic map from $\bar{X}$ to $\mathbb{C}P^1$. This map is lifted to $\mathbb{L}$ a locally univalent meromorphic function $W : \mathbb{L} \to \mathbb{C}P^1$ called the *developing map* of this $\mathbb{C}P^1$-structure. It is uniquely determined by the $\mathbb{C}P^1$-structure up to post-composition by Möbius transformations.

When we take an analytic continuation of a local coordinate function along a closed curve on $\bar{X}$ and come back to the initial point, it differs from the previous one by a Möbius transformation since the transition maps are in $PSL_2(\mathbb{C})$. Consequently we have a homomorphism $\chi : \Gamma \cong \pi_1(\bar{X}) \to PSL_2(\mathbb{C})$ which is called the *holonomy representation* and satisfies $\chi(\gamma) \circ W = W \circ \gamma$ for all $\gamma \in \Gamma$. Therefore the $\mathbb{C}P^1$-structure on $\bar{X}$ determines the pair (W, χ) up to the action of $PSL_2(\mathbb{C})$ and vice versa. Here we show the most basic example of $\mathbb{C}P^1$-structures on $\bar{X}$: Let W be the identity map $W : \mathbb{L} \hookrightarrow \mathbb{C}P^1$ and χ also be the identity homomorphism $\chi : \Gamma \hookrightarrow PSL_2(\mathbb{R})$ which induces a local coordinate function as a local inverse of the universal covering map $\mathbb{L} \to \bar{X}$. We call this $\mathbb{C}P^1$-structure the *standard $\mathbb{C}P^1$-structure on $\bar{X}$*.

Let $P(\bar{X}) = \{(W, \chi)\}/PSL_2(\mathbb{C})$ be the set of $\mathbb{C}P^1$-structures on $\bar{X}$. We will parametrize $P(\bar{X})$ by holomorphic quadratic differentials on $\bar{X}$ as follows: A holomorphic function φ on $\mathbb{L}$ is called a *holomorphic quadratic differential* for Γ if it satisfies

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z)$$

for all $z \in \mathbb{L}$ and $\gamma \in \Gamma$. It is a lift of holomorphic quadratic differentials on $\bar{X} = \mathbb{L}/\Gamma$. Let $Q(\bar{X})$ be the set of holomorphic quadratic differentials for Γ whose hyperbolic sup norm $||\varphi|| = \sup_{z \in \mathbb{L}} |\Im z|^2 |\varphi(z)|$ is bounded. $Q(\bar{X})$ has a structure of complex linear space of $dim_{\mathbb{C}} Q(\bar{C}) = 3g - 3 + n$ which is equal to the dimension of $T(X)$. We show that there is a canonical bijection between $P(\bar{X})$ and $Q(\bar{X})$ which maps the standard $\mathbb{C}P^1$-structure to the origin: Given a $\mathbb{C}P^1$-structures on $\bar{X}$, take the *Schwarzian derivative* of W

$$S_W := (f''/f')' - \frac{1}{2}(f''/f')^2$$

which is an element of $Q(\bar{X})$. Conversely given a holomorphic quadratic differential φ for Γ, solve the differential equation $S_f = \varphi$ on $\mathbb{L}$. In practice to find the solution f, we consider the following linear homogeneous ordinary differential equation of the second order

$$2\eta'' + \varphi\eta = 0$$

on $\mathbb{L}$. Since $\mathbb{L}$ is simply connected, a unique solution η exists on $\mathbb{L}$ for the given initial data $\eta(-i) = a$ and $\eta'(-i) = b$. Let η_1 and η_2 be the solution defined by the conditions $\eta_1(-i) = 0$ and $\eta_1'(-i) = 1$, and $\eta_2(-i) = 1$ and $\eta_2'(-i) = 0$. Then the ratio $f_\varphi = \eta_1/\eta_2$ is a locally univalent meromorphic function on $\mathbb{L}$, the developing map associated with φ. A direct computation shows that $\eta(\gamma(z))(\gamma'(z))^{-\frac{1}{2}}$ also satisfies the above equation hence there is a matrix of $SL_2(\mathbb{C})$ such that

$$\begin{pmatrix} \eta_1(\gamma(z))(\gamma'(z))^{-\frac{1}{2}} \\ \eta_2(\gamma(z))(\gamma'(z))^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

for all $\gamma \in \Gamma$. As a result we have a homomorphism $\chi_\varphi : \Gamma \to PSL_2(\mathbb{C})$, the holonomy representation associated with φ. We can also consider χ_φ as the monodromy representation of the above differential equation.

5 Bers embedding of $T(X)$

Now we embed $T(X)$ into $Q(\bar{X}) \cong \mathbb{C}^{3g-3+n}$ by means of the identification $P(\bar{X}) \cong Q(\bar{X})$. For each element $[\mu] \in T(X) = B_1(\Gamma, \mathbb{H})/\sim$, $f^\mu|_{\mathbb{L}}$ is conformal and $\Gamma^\mu = f^\mu\Gamma(f^\mu)^{-1}$ is a quasi-fuchsian group. Therefore it determines a $\mathbb{CP}^1$-structure on $\mathbb{L}/\Gamma$ where the developing map is $W = f^\mu|_{\mathbb{L}}$ and the holonomy representation $\chi : \Gamma \to \Gamma^\mu$ is defined by $\chi(\gamma) = f^\mu\gamma(f^\mu)^{-1}$. After the identification $P(\bar{X}) \cong Q(\bar{X})$, $T(X)$ can be embedded into $Q(\bar{X})$, which is called the *Bers embedding* of $T(X)$.

We will show not only the picture of $T(X)$ but also other $\mathbb{C}P^1$-structures on $\bar{X}$: Let $K(\bar{X})$ be the set of $\mathbb{C}P^1$-structures on $\bar{X}$ whose holonomy groups are Kleinian groups, discrete subgroups of $PSL_2(\mathbb{C})$. Shiga [4] showed that the connected component of the interior of $K(\bar{X})$ containing the origin coincides with $T(X)$. Shiga and Tanigawa [5] proved that any $\mathbb{C}P^1$-structure of the interior of $K(\bar{X})$ has a quasi-fuchsian holonomy representation. Nehari showed that $T(X)$ is bounded in $Q(\bar{X})$ with respect to the hyperbolic sup norm $||\varphi|| = \sup_{z\in\mathbb{L}} |\Im z|^2|\varphi(z)|$, while Tanigawa proved that $K(\bar{X})$ is unbounded.

6 Pictures of $T(X)$ and $K(X)$

We will show pictures of $T(X)$ and $K(X)$, all of which depends on the underlying complex structure of $\bar{X}$. All picture were drawn by Yasushi Yamashita. Figure 1 and figure 2 are the case that $\bar{X}$ has a hexagonal symmetry. Figure 3 and figure 4 are the case that $\bar{X}$ has a square symmetry. Black colored region consists of φ whose holonomy representation has an indiscrete image. For both cases, $T(X)$ looks like an isolated planet, while $K(X)$ itself looks like the galaxy: Some planets seem to bump each other... When we take $\bar{X}$ anti-symmetric, $T(X)$ and $K(X)$ become distorted, which we can see in figure 5 and figure 6.

To draw these pictures we need

1. to calculate the holonomy representation χ_φ for $\varphi \in Q(\bar{X})$, and
2. to check whether $\chi_\varphi(\Gamma)$ is discrete or not.

First we will explain (1). To determine χ_φ, we must solve $\mathcal{S}_f = \varphi$ on $\mathbb{L}$. In general $\varphi \in Q(\bar{X})$ is highly transcendental function on $\mathbb{L}$ and it is very difficult for us to handle it. Here is an idea: If $dim_{\mathbb{C}} T(X) = 3g - 3 + n = 1$, then $(g, n) = (0, 4)$ or $(1, 1)$. Take $\bar{X} = \mathbb{CP}^1 - \{0, 1, \infty, \lambda\}$, then we can find a basis of $Q(\bar{X})$ like $Q(\bar{X}) = \mathbb{C} \cdot \pi^*(\frac{1}{w(w-1)(w-\lambda)})$. Even in this case, it is still difficult to solve

$$\mathcal{S}_f = \pi^*(\frac{t}{w(w-1)(w-\lambda)})$$

where $\pi : \mathbb{L} \to \mathbb{CP}^1 - \{0, 1, \infty, \lambda\}$ and $t \in \mathbb{C} \cong Q(\bar{X})$. But we can push down the above equation onto $\bar{X} = \mathbb{CP}^1 - \{0, 1, \infty, \lambda\}$

$$\mathcal{S}_{f \circ \pi^{-1}} = \frac{t}{w(w-1)(w-\lambda)} + (\frac{1}{2w^2(w-1)^2} + \frac{1}{2(w-\lambda)^2} + \frac{c(\lambda)}{w(w-1)(w-\lambda)})$$

where $c(\lambda)$ is called the *accessory parameter* of $\pi : \mathbb{L} \to \bar{X}$.

To get the solution we take the ratio of two linearly independent solution of

$$2y'' + (\frac{1}{2w^2(w-1)^2} + \frac{1}{2(w-\lambda)^2} + \frac{t + c(\lambda)}{w(w-1)(w-\lambda)})y = 0$$

and calculate the monodromy group of this equation with respect to closed paths of $\pi_1(\bar{X}) \cong F_3$. Since the above ordinary differential equation has rational coefficients on $\mathbb{C}P^1$, we can use computer to get the image of 3 generators of $\pi_1(\bar{X})$ in $PSL_2(\mathbb{C})$ numerically. Here we remark that to draw the picture of $K(X)$ up to parallel translation, we don't need to determine the accessory parameter $c(\lambda)$ in practice.

For (2), we apply Shimizu lemma to check whether $\chi_\varphi(\Gamma)$ is indiscrete, and Poincaré theorem to construct the Ford fundamental domain to check whether $\chi_\varphi(\Gamma)$ is discrete. This part is so called Jorgensen theory and has been proved recently by Akiyoshi, Sakuma, Wada and Yamashita [1].

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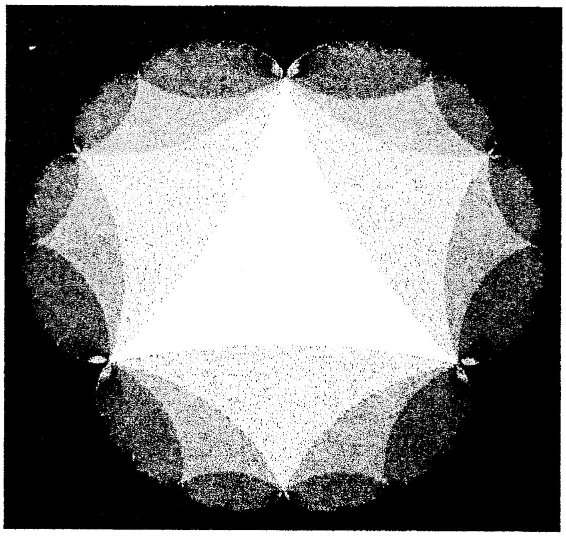

Figure 1: $T(X)$ for hexagonal symmetry

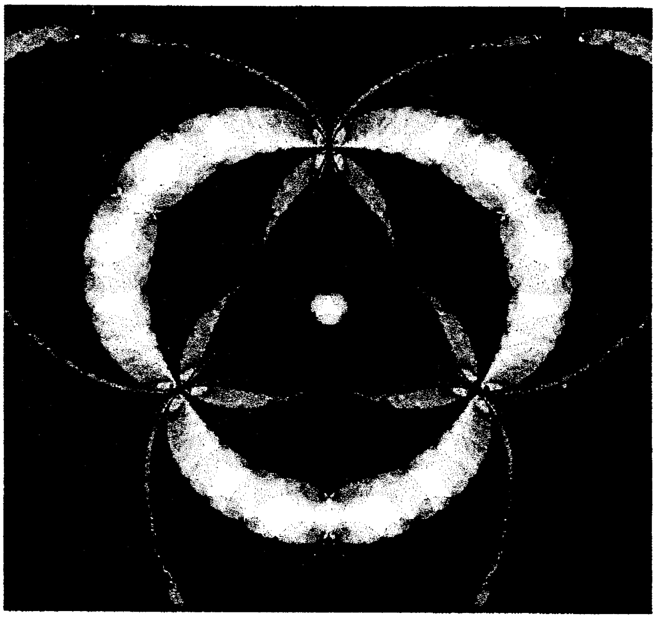

Figure 2: $K(X)$ for hexagonal symmetry

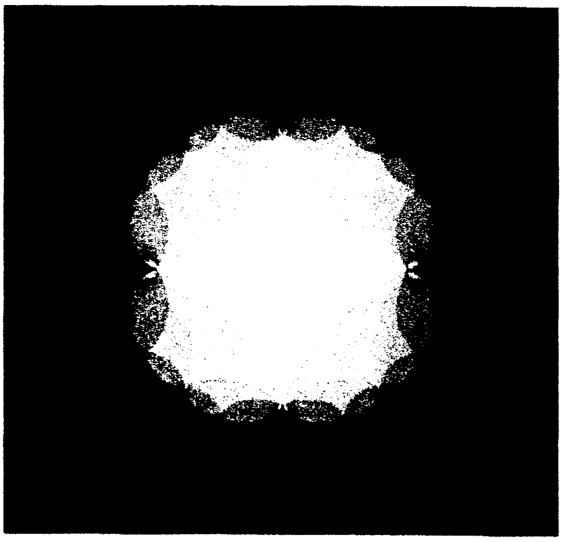

Figure 3: $T(X)$ for square symmetry

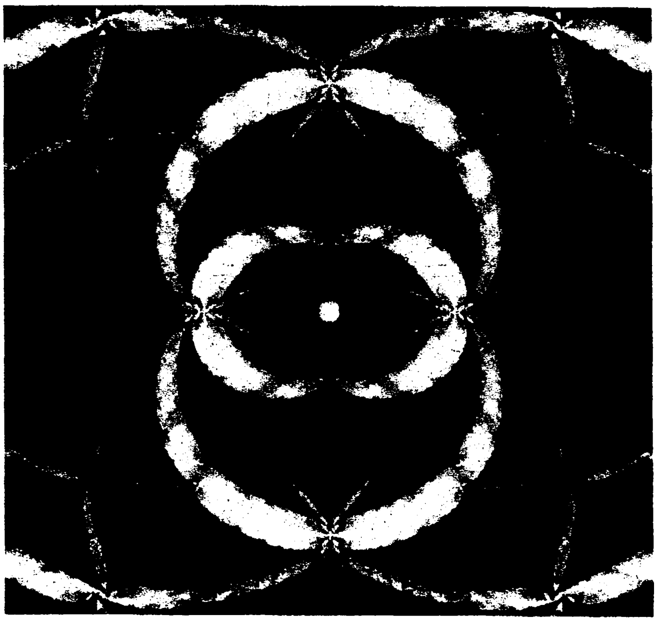

Figure 4: $K(X)$ for square symmetry

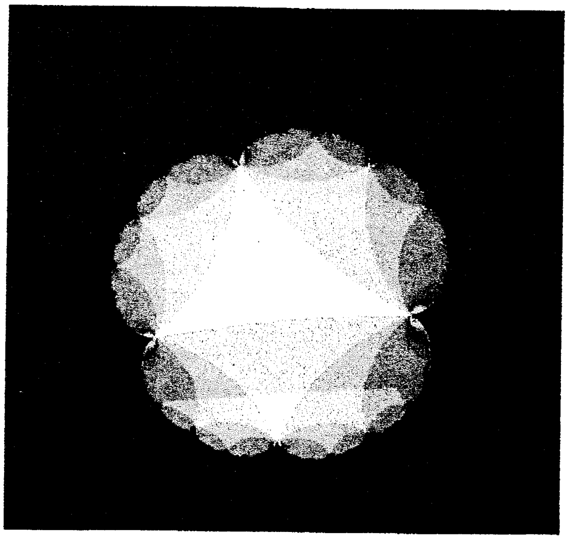

Figure 5: distorted $T(X)$

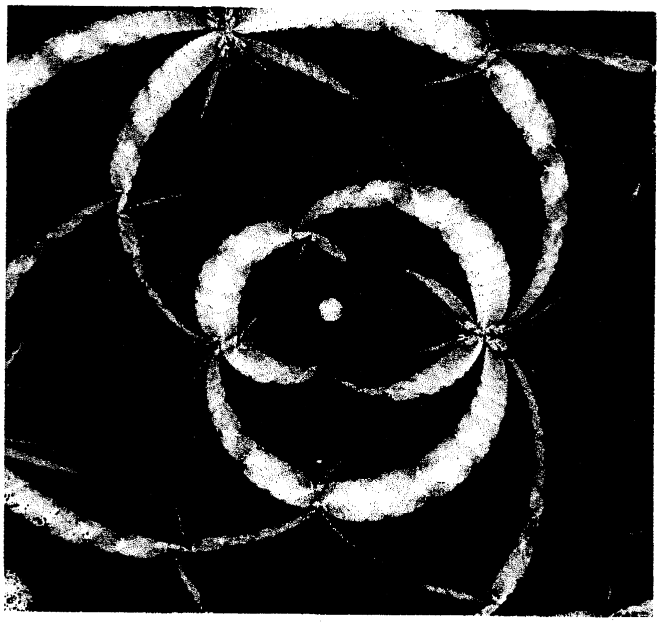

Figure 6: distorted $K(X)$